MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME: Mr. Lim

CENTRE NUMBER: S

INDEX NUMBER:

MATHEMATICS

9758/01

Paper 1

October/November 2022

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Solution served as a suggestion only

1 **Do not use a calculator in answering this question.**

The complex numbers z and w satisfy the following equations.

$$iz + 2w = -1$$

$$(2 - i)z + iw = 6$$

Find z and w, giving your answers in the form $a + ib$ where a and b are real numbers. [4]

$-z + 2iw = -i$

$2iw = z - i$

$iw = \frac{1}{2}(z - i)$

sub in

$(2-i)z + \frac{1}{2}(z - i) = 6$

$(2 - i + \frac{1}{2})z = 6 + \frac{1}{2}i$

$(\frac{5}{2} - i)z = 6 + \frac{1}{2}i$

$(5 - 2i)z = 12 + i$

$z = \frac{12+i}{5-2i} \times \frac{5+2i}{5+2i} = 2+i$

$\Rightarrow 2i - 1 + 2w = -1$

$w = -i$ //

2 It is given that $f(x) = \tan^{-1}(\sqrt{2} + x)$.

(a) Find $f'(x)$ and $f''(x)$. [3]

$$f'(x) = \frac{1}{1+(\sqrt{2}+x)^2}$$

$$= \frac{1}{1+(2+2\sqrt{2}x+x^2)}$$

$$= \frac{1}{3+2\sqrt{2}x+x^2}$$

$$f'(0) = \frac{1}{3}$$

$$f''(x) = -\frac{2\sqrt{2}+2x}{(3+2\sqrt{2}x+x^2)^2}$$

$$f''(0) = -\frac{2\sqrt{2}}{9}$$

(b) Hence find the first three terms of the Maclaurin series for $f(x)$. Give the coefficients correct to 3 significant figures. [3]

$$f(0) = 0.955$$

$$f(x) = 0.955 + 0.333x - 0.157x^2 + \ldots$$

3 The parametric equations of a curve are $x = \frac{1}{2}(e^{3t} + 2e^{-3t})$ and $y = \frac{1}{2}(e^{3t} - 2e^{-3t})$.

(a) Using calculus, find the gradient of the normal to the curve at the point where $t = \frac{1}{3}\ln 2$. [3]

$\frac{dx}{dt} = \frac{1}{2}\left(3e^{3t} - 6e^{-3t}\right)$ $\qquad$ $\frac{dy}{dt} = \frac{1}{2}\left(3e^{3t} + 6e^{-3t}\right)$

$\frac{dy}{dx} = \frac{3e^{3t} + 6e^{-3t}}{3e^{3t} - 6e^{-3t}}$

sub in $t = \frac{1}{3}\ln 2$

$\frac{dy}{dx} = \frac{6+3}{6-3} = 3$

$\therefore$ Grad of normal $= -\frac{1}{3}$

(b) By considering x^2 and y^2 or otherwise, find the cartesian equation of the curve, stating any restriction on the values of x. [3]

$x^2 = \frac{1}{4}(e^{3t} + 2e^{-3t})(e^{3t} + 2e^{-3t})$ $\quad y^2 = \frac{1}{4}(e^{6t} - 4 + 4e^{-6t})$

$= \frac{1}{4}(e^{6t} + 4 + 4e^{-6t})$

$\Rightarrow \quad x^2 - y^2 = 2$

using G.C , $x \geqslant \sqrt{2}$

4 **(a)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x$. [2]

$$\frac{d}{dx}\frac{1}{\tan x} = \frac{0 - \sec^2 x}{\tan^2 x}$$

$$= \frac{-\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}$$

$$= -\operatorname{cosec}^2 x$$

(b) Show that $\sin 2x \tan x = 2\sin^2 x$. [1]

LHS: $2\sin x\cos x\left(\frac{\sin x}{\cos x}\right)$

$= 2\sin^2 x$ //

(c) Hence, find the exact value of $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec} 6x \cot 3x \, dx$. [4]

$$\operatorname{cosec} 6x \cot 3x = \frac{1}{\sin 6x \tan 3x}$$

$$= \frac{1}{2\sin^2 3x}$$

$$\Rightarrow \int_{\pi/18}^{\pi/9} \operatorname{cosec} 6x \cot 3x \, dx = \int_{\frac{\pi}{18}}^{\pi/9} \frac{1}{2\sin^2 3x} \, dx$$

$$= -\frac{1}{2}\int_{\pi/18}^{\pi/9} -\operatorname{cosec}^2 3x \, dx$$

$$= -\frac{1}{2}\left[\frac{1}{3}\cot 3x\right]_{\pi/18}^{\pi/9}$$

$$= -\frac{1}{6}\left[\frac{1}{\sqrt{3}} - \sqrt{3}\right]$$

$$= -\frac{1}{6}\left(\frac{\sqrt{3} - 3\sqrt{3}}{3}\right)$$

$$= \frac{\sqrt{3}}{9}$$

5 The line with equation $y = mx$ is a tangent to the curve with equation

$$(x+8)^2 + (y-14)^2 = 52.$$

(a) Show that m satisfies the equation

$$3m^2 + 56m + 36 = 0.$$

[4]

$x^2 + 16x + 64 + (mx - 14)^2 = 52$

$x^2 + 16x + m^2x^2 - 28mx + 196 = -12$

$(m^2+1)x^2 + (16 - 28m)x + 208 = 0$

$b^2 - 4ac = 0$

$(16-28m)^2 - 4(m^2+1)(208) = 0$

$256 - 896m + 784m^2 - 832m^2 - 832 = 0$

$-48m^2 - 896m - 576 = 0$

$3m^2 + 56m + 36 = 0$

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A and B are points on the curve. The tangent at A and the tangent at B intersect at the origin.

(b) Find the coordinates of A and B. [4]

$$3m^2 + 56m + 36 = 0$$

$$m = -\frac{2}{3} \text{ or } m = -18$$

sub in $(m^2+1)x^2 + (16-28m)x + 208 = 0$

$$\frac{13}{9}x^2 + \frac{104}{3}x + 208 = 0$$

$$x = -12 \quad , \quad y = 8$$

sub in $(m^2+1)x^2 + (16-28m)x + 208 = 0$

$$325x^2 + 520x + 208 = 0$$

$$x = -\frac{4}{5} \quad y = \frac{72}{5}$$

$\therefore$ coordinates of A & B are $(-12, 8)$ and $\left(-\frac{4}{5}, \frac{72}{5}\right)$

6 The function f is defined by

$$f : x \to \frac{ax+k}{x-a}, \quad x \in \mathbb{R}, \quad x \neq a$$

where a and k are constants.

(a) Describe fully a sequence of transformations which transforms the curve $y = \frac{1}{x}$ onto the curve $y = f(x)$. [4]

let $y = \frac{ax+k}{x-a}$

$= a + \frac{k+a^2}{x-a}$

$y = \frac{1}{x}$

(1) Translate a units in positive x-axis direction

$y = \frac{1}{x-a}$

(2) scaling parallel to y-axis by factor of $k+a^2$

$y = \frac{k+a^2}{x-a}$

(3) Translate a units in positive y-axis direction

$y = a + \frac{k+a^2}{x-a}$

(b) Find $f^{-1}(x)$. [2]

let $y = \frac{ax+k}{x-a}$

$xy - ay = ax + k$

$xy - ax = k + ay$

$x(y-a) = k + ay$

$x = \frac{k+ay}{y-a}$

$f^{-1}(x) = \frac{k+ax}{x-a}$, $x \in \mathbb{R}$, $x \neq a$

(c) Hence, or otherwise, find $f^2(x)$. [1]

$f^2(x) = ff^{-1}(x)$

$= x$

(d) Find $f^{2023}(1)$ in terms of a and k. [2]

$f^{2023}(1) = f(1)$

$= \frac{k+a}{1-a}$

7 A curve C has equation $y = x^{-3}\ln x$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{1-3\ln x}{x^4}$ and hence find the coordinates of the turning point of C. [4]

$$y = x^{-3}\ln x$$

$$\frac{dy}{dx} = -3x^{-4}\ln x + x^{-4}$$

$$= \frac{1-3\ln x}{x^4}$$

$\frac{dy}{dx} = 0$ $\quad 3\ln x = 1$

$$\ln x = \frac{1}{3}$$

$$x = e^{1/3} \qquad y = e^{-1}\cdot\frac{1}{3}$$

$$= \frac{1}{3e}$$

$\therefore$ coordinate $\left(e^{\frac{1}{3}}, \frac{1}{3e}\right)$

(b) Find the exact area enclosed by C, the x-axis and the line $x = 3$. [5]

$$\int_1^3 x^{-3} \ln x \, dx$$

$u = \ln x$ $\qquad \frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = -\frac{1}{2}x^{-2}$

$$= \left[-\frac{1}{2}x^{-2}\ln x\right]_1^3 + \int_1^3 \frac{1}{2}x^{-3}\,dx$$

$$= \left[-\frac{1}{18}\ln 3\right] + \left[-\frac{1}{4}x^{-2}\right]_1^3$$

$$= -\frac{1}{18}\ln 3 + \left(-\frac{1}{36} + \frac{1}{4}\right)$$

$$= \frac{2}{9} - \frac{1}{18}\ln 3$$

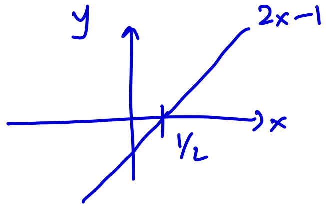
8 (a) Find $\int \frac{2x-1}{x^2+2x+1}\,dx$. [4]

$= \int \frac{2x+2}{x^2+2x+1} - \frac{3}{(x+1)^2}\,dx$

$= \ln|x^2+2x+1| + \frac{3}{x+1} + c$

$= 2\ln|x+1| + \frac{3}{x+1} + c$

(b) Find the exact value of $\displaystyle\int_0^2 \frac{|2x-1|}{x^2+2x+1}\,\mathrm{d}x$. [3]



$$= \int_0^{1/2} -\frac{2x-1}{x^2+2x+1}\,dx + \int_{1/2}^{2} \frac{2x-1}{x^2+2x+1}\,dx$$

$$= -\left[2\ln|x+1| + \frac{3}{x+1}\right]_0^{1/2} + \left[2\ln|x+1| + \frac{3}{x+1}\right]_{1/2}^{2}$$

$$= -\left[2\ln\tfrac{3}{2} + 2 - 3\right] + \left[2\ln 3 + 1 - 2\ln\tfrac{3}{2} - 2\right]$$

$$= -4\ln\tfrac{3}{2} + 2\ln 3$$

$$= -4(\ln 3 - \ln 2) + 2\ln 3$$

$$= 4\ln 2 - 2\ln 3$$

9 (a) An arithmetic series has first term a and common difference d, where $d \neq 0$. The first, thir fifteenth terms of this series are the first, second and third terms of a geometric series. Finc terms of a.

$$\frac{a+2d}{a} = \frac{a+14d}{a+2d}$$

$$a^2+4ad+4d^2 = a^2+14ad$$

$$-10ad+4d^2=0$$

$$d(-10a+4d)=0$$

$\because d \neq 0 \qquad 10a = 4d$

$$d = \frac{5}{2}a$$

(b) A geometric series has first term $\sin\theta$ and common ratio $-\cos\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that the sum to infinity of this series is $\tan k\theta$, where k is a constant to be found. [3]

$$S_\infty = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{1+2\cos^2\frac{1}{2}\theta-1}$$

$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$= \tan\frac{1}{2}\theta \qquad \therefore k=\frac{1}{2}$$

(ii) Given that $\theta = \frac{\pi}{3}$, find the exact sum of the first seven terms of this series. [2]

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\qquad -\cos\frac{\pi}{3} = -\frac{1}{2}$

$$S_7 = \frac{\frac{\sqrt{3}}{2}\left(1-\left(-\frac{1}{2}\right)^7\right)}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2}\left(1+\frac{1}{2^7}\right)}{\frac{3}{2}}$$

$$= \frac{\sqrt{3}}{3}\left(1+\frac{1}{2^7}\right)$$

$$= \frac{43\sqrt{3}}{128}$$

10 A curve C has equation $y = ax + b + \frac{a+2b}{x-1}$, where a and b are real constants such that $a > 0$, $b \neq -\frac{1}{2}a$ and $x \neq 1$.

(a) Given that C has no stationary points, use differentiation to find the relationship between a and b. [3]

$$\frac{dy}{dx} = a - (a+2b)(x-1)^{-2}$$

$$\frac{dy}{dx} = 0 \qquad a = \frac{a+2b}{(x-1)^2}$$

$$(x-1)^2 = \frac{a+2b}{a}$$

$$x^2 - 2x + 1 - \frac{a+2b}{a} = 0$$

$\because$ No turning point

$$b^2 - 4ac < 0$$

$$4 - 4(1)\left(1 - \frac{a+2b}{a}\right) < 0$$

$$1 - 1 + \frac{a+2b}{a} < 0$$

$$a + 2b < 0$$

$$a < -2b$$

It is now given that $b = -2a$. $y = ax - 2a - \frac{3a}{x-1}$

(b) Sketch C on the axes on page 19 stating the equations of any asymptotes and the coordinates of the points where C crosses the axes. [4]

when $x = 0$ $\quad y = -2a + 3a$

$$= a$$

when $y = 0$ $\quad ax - 2a = \frac{3a}{x-1}$

$$(x-2)(x-1) = 3$$

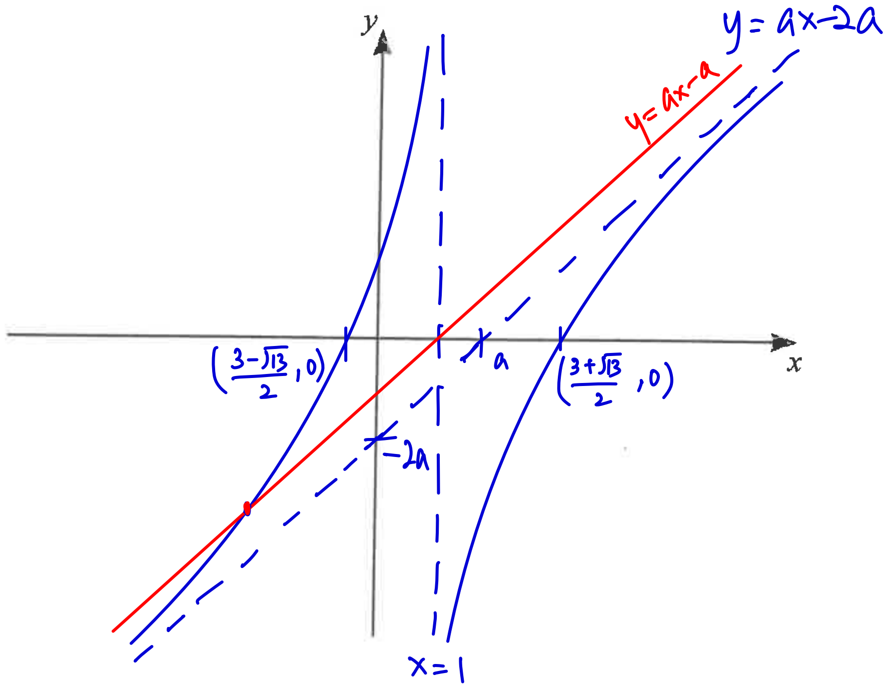
$$x^2 - 3x - 1 = 0$$

$$x : \frac{3 \pm \sqrt{9 - 4(1)(-1)}}{2}$$

$$x : \frac{3 \pm \sqrt{13}}{2}$$

asymptotes $x = 1$, $y = ax - 2a$

$\because$ $b = -2a$, curve has no stationary point



(c) On the same axes, sketch the graph of $y = ax - a$. [1]

(d) Hence solve the inequality $x - 2 - \frac{3}{x-1} \leqslant x - 1$. [2]

$$a\left(x-2-\frac{3}{x-1}\right) \leq a(x-1) \quad \because a>0$$

$$ax-2a-\frac{3a}{x-1} \leq ax-a$$

$$-a-\frac{3a}{x-1} \leq 0$$

$$1-\frac{3}{x-1} \leq 0$$

From graph: $x \leq -2$ or $x > 1$

11 A gas company has plans to install a pipeline from a gas field to a storage facility. One part of the route for the pipeline has to pass under a river. This part of the pipeline is in a straight line between two points, P and Q.

Points are defined relative to an origin $(0, 0, 0)$ at the gas field. The x-, y- and z-axes are in the directions east, north and vertically upwards respectively, with units in metres. P has coordinates $(1136, 92, p)$ and Q has coordinates $(200, 20, -15)$.

(a) The length of the pipeline PQ is 939 m. Given that the level of P is below that of Q, find the value of p. [3]

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} -936 \\ -72 \\ -15-p \end{pmatrix}$$

$$|\overrightarrow{PQ}| = 939 = \sqrt{(-936)^2 + (-72)^2 + (-15-p)^2}$$

$$441 = 225 + 30p + p^2$$

$$p^2 + 30p - 216 = 0$$

$$p = 6 \quad \text{or} \; -36 \qquad \because \text{point } P \text{ is below } Q, \; p = -36$$

A thin layer of rock lies below the ground. This layer is modelled as a plane. Three points in this plane are $(400, 600, -20)$, $(500, 200, -70)$ and $(600, -340, -50)$.

(b) Find the cartesian equation of this plane. [4]

$$\text{let } \overrightarrow{OA} = \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} \quad \overrightarrow{OB} = \begin{pmatrix} 500 \\ 200 \\ -70 \end{pmatrix} \quad \overrightarrow{OC} = \begin{pmatrix} 600 \\ -340 \\ -50 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 100 \\ -400 \\ -50 \end{pmatrix} = -50\begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} 100 \\ -540 \\ 20 \end{pmatrix} = 20\begin{pmatrix} 5 \\ -27 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix} \times \begin{pmatrix} 5 \\ -27 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8+27 \\ -(-2-5) \\ 54-40 \end{pmatrix} = \begin{pmatrix} 35 \\ 7 \\ 14 \end{pmatrix} = 7\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 2560 \qquad \therefore \; \mathbf{r} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 2560$$

$$5x + y + 2z = 2560$$

(c) Hence find the coordinates of the point where the pipeline meets the rock. [4]

$$\vec{PQ} = \begin{pmatrix} -936 \\ -72 \\ 21 \end{pmatrix} = 3\begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix}$$

line PQ : $r = \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix}$ where $\lambda \in \mathbb{R}$

$$\begin{pmatrix} 200+312\lambda \\ 20+24\lambda \\ -15-7\lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 2560$$

$$990 + 1570\lambda = 2560$$

$$\lambda = 1$$

$\therefore$ pt of intersection : $\begin{pmatrix} 512 \\ 44 \\ -22 \end{pmatrix}$

$\therefore$ coordinate : $(512, 44, -22)$

(d) Find the angle that the pipeline between the points P and Q makes with the horizontal. [2]

let $\measuredangle$ be θ

$$\sin\theta = \frac{\left|\begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}{\left|\begin{matrix} 312 \\ 24 \\ -7 \end{matrix}\right|}$$

$$\sin\theta = \frac{7}{313}$$

$$\theta = 1.28^\circ$$

$$\approx 1.3^\circ$$

12 Scientists are interested in the population of a particular species. They attempt to model the population P at time t days using a differential equation. Initially the population is observed to be 50 and after 10 days the population is 100.

The first model the scientists use assumes that the rate of change of the population is proportional to the population.

(a) Write down a differential equation for this model and solve it for P in terms of t. [5]

$$\frac{dP}{dt} = kP$$

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$$\ln|P| = kt + c$$

$$P = Ae^{kt} \text{ where } A = \pm e^{c}$$

when $t=0$ $P=50$

$$\therefore A = 50$$

$$\Rightarrow P = 50e^{kt}$$

when $t=10$, $P=100$

$$2 = e^{k(10)}$$

$$\ln 2 = 10k$$

$$k = \frac{1}{10}\ln 2$$

$$\Rightarrow P = 50e^{(\frac{1}{10}\ln 2)t}$$

$$= 50\left(2^{\frac{t}{10}}\right)$$

To allow for constraints on population growth, the model is refined to

$$\frac{dP}{dt} = \lambda P(500 - P)$$

where λ is a constant.

(b) Solve this differential equation to find P in terms of t. [6]

$$\int \frac{1}{P(500-P)}\, dP = \int \lambda\, dt$$

$$\int \frac{1}{-P^2+500P}\, dP = \int \lambda\, dt$$

$$\int \frac{1}{-(P-250)^2+250^2}\, dP = \lambda t + C$$

$$\frac{1}{500}\ln\left|\frac{P}{500-P}\right| = \lambda t + C$$

$$\frac{P}{500-P} = Ae^{500\lambda t} \quad \text{where } A = \pm e^{500C}$$

when $t=0$ $P=50$

$$\therefore \frac{1}{9} = A$$

when $P=100$, $t=10$

$$\ln\frac{9}{4} = 5000\lambda$$

$$\lambda = \frac{1}{5000}\ln\frac{9}{4}$$

$$\frac{P}{500-P} = \frac{1}{9}e^{\left(\frac{1}{10}\ln\frac{9}{4}\right)t}$$

$$= \frac{1}{9}\left(\frac{9}{4}\right)^{\frac{t}{10}}$$

$$9P\left(\frac{9}{4}\right)^{-\frac{t}{10}} = 500 - P$$

$$P\left(1+9\left(\frac{9}{4}\right)^{-t/10}\right) = 500$$

$$P = \frac{500}{1+9\left(\frac{9}{4}\right)^{-t/10}}$$

\#

Question 12 continues on the next page.

Solution served as a suggestion only

(c) Using the refined model, state the population of this species in the long term. Comment on how this value suggests the refined model is an improvement on the first model. [2]

$$P = \frac{500}{1 + 9\left(\frac{9}{4}\right)^{-t/10}}$$

$$t \to \infty, \quad \left(\frac{9}{4}\right)^{-t/10} \to 0$$

$$\therefore P \to 500$$

For the first model, population will grow infinitely in long run and the refined model has a limit at 500, which is more realistic and thus, it is an improvement